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Non-propagating degrees of freedom in supergravity and very extended G_2

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Abstract: Recently a correspondence between non-propagating degrees of freedom in maximal supergravity and the very extended algebra E_{11} has been found. We perform a similar analysis for a supergravity theory with eight supercharges and very extended G_2. In particular, in the context of $d = 5$ minimal supergravity, we study whether supersymmetry can be realised on higher-rank tensors with no propagating degrees of freedom. We find that in this case the very extended algebra fails to capture these possibilities.

Keywords: Global Symmetries, Supersymmetry and Duality, Supergravity Models.

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Contents

1. Introduction

The interplay between supergravities and their associated Kac-Moody algebras has received a great amount of attention over the years.

An important first step was the discovery of hidden symmetries [1, 2] upon reduction to lower dimensions. In three dimensions, one obtains gravity coupled to a scalar coset G/H. Further reduction to two dimensions leads to a symmetry which is the affine extension G^+, analogous to the Geroch group $\mathrm{SL}(2,\mathbb{R})^+$ for pure gravity [3 − 5]. In one dimension the relevant symmetry is expected to be the over extension G^{++} [3, 6]. The latter has mainly been considered in the context of eleven-dimensional supergravity near space-like singularities and $E_{10} = E_8^{++}$ [7, 8], see [9] for IIB. In this framework, space-time is expected to arise from the dynamics of a σ-model in one dimension. In addition, there is a conceptually different approach based on the non-linear realisation of (the conformal group together with) the very extension[1] G^{+++}, like $E_{11} = E_8^{+++}$ for the $d = 11$ theory [11, 12] as well as the IIB theory [13].

Very recently, the relation between the non-propagating degrees of freedom of supergravity, closure of the supersymmetry algebra and the corresponding Kac-Moody algebras has come into focus. In particular, in [14, 15] it was shown how all the mass deformations and possible gaugings of maximal supergravity in $d \geq 3$ dimensions,[2] or rather the $(d-1)$-forms dual to these constants, correspond to specific generators in the very extended algebra E_{11}. An exception must be made here for gaugings that violate the action principle, as will also be discussed in section 4. In addition, E_{11} makes predictions for the possible multiplets for the d-forms on which the superalgebra can be realised [9, 17]. Although these

[1] The simultaneous non-linear realization of the affine group and the conformal group in four dimensions reproduces the Einstein equation of general relativity [10].

[2] It would be interesting to see if the recent results on gaugings in $d = 2$ of [16] can be incorporated in E_{11} as well.

forms do not carry any propagating degrees of freedom, they are part of the field content of the theory and play a crucial role in the story of space-time filling branes [18, 19]. The possible d-forms that are allowed by the superalgebra have been explicitly calculated in the cases of IIB [20] and IIA [21] and found to agree with the E_{11} predictions.

The philosophy underlying the recent papers [14, 15] can be summarised as follows. Given any very extended algebra, one can decompose its adjoint representation into representations of a Lie subalgebra SL(d) (the 'gravity line'). These are labelled by their level l in the Kac-Moody algebra. Up to some level l, the resulting generators are interpreted as the d-dimensional space-time fields of the corresponding supergravity. Generators at higher level are interpreted as space-time fields with more than d indices, and these may correspond to dual formulations of lower-level fields or non-propagating degrees of freedom [22]. Given this dictionary, one can read off the possible $(d-1)$- and d-forms for any very extended algebra and compare this to the closure of the supersymmetry algebra on such forms. This extends the results of [23] for propagating degrees of freedom to the non-propagating $(d-1)$- and d-forms.

The previous ideas have also been applied to less than very extended algebras. For example, the propagating degrees of freedom of supergravity theories can likewise be obtained from the affinely extended G^{+}, see e.g. [24] for a detailed account. In addition, the overextended algebras G^{++} can contain generators corresponding to the $(d-1)$-forms. An example in $d=10$ for the overextended E_{10} can be found in [25]. However, only the very extended G^{+++} may capture all non-propagating degrees of freedom. Roughly speaking, the less than very extended algebras seem to be 'too small' to contain both $(d-1)$- and d-forms.

An interesting question is whether it is possible to extend the striking results for E_{11} to cases based on other very extended algebras. In other words, do other very extended algebras also predict the correct $(d-1)$- and d-forms for the associated supergravity theory in d dimensions? This will necessarily be in the context of supergravities with less than maximal supersymmetry (as maximal supergravities are associated to E_{11}), while the very extended algebras will be based on other Lie algebras G than the most exceptional E_8. Hence all other cases are far less restricted by symmetries. An obvious and worthwhile question is whether the correspondence found for E_{11} also holds for these less symmetric situations and if not, what the requirements are for it to hold or what the reasons of its failure are.

In this note we address this question in the context of minimal $N=2$ pure supergravity in $d=5$. This theory is similar to $d=11$ supergravity in a number of respects, see e.g. [26]: for instance, its bosonic field content only contains a metric and a $(d-2)/3$-form A with Chern-Simons term $A \wedge dA \wedge dA$. Clear differences are that it has only 8 instead of 32 supercharges, and it reduces to the coset $G_2/\,\mathrm{SO}(4)$ in three dimensions, see e.g. [27]. Hence the relevant very extended algebra is G_2^{+++} instead of E_{11}. It is of interest to see whether this affects the correspondence between the non-propagating degrees of freedom and the very extended algebra. To this end we first consider the supersymmetry algebra of this theory and see on which $(d-1)$ and d-forms this can be realised. It turns out that the allowed $(d-1)$-forms transform as a triplet under the SU(2) R-symmetry. Afterwards we compare this with the predictions from very extended G_2 and finish with a discussion of our results.

2. Minimal supergravity in $d = 5$

We use the conventions of [28, 29]. Our metric is mostly plus. Curved (flat) indices are denoted by Greek (Latin) letters $\mu, \nu \ldots$ $(m, n, \ldots)$. The index $i = 1, 2$ labels the two symplectic anti-commuting fermions and is raised and lowered according to $\psi^i_\mu = \varepsilon^{ij}\psi_{\mu j}$ and $\psi_{\mu j} = \psi^i_\mu \varepsilon_{ij}$ with $\varepsilon_{12} = \varepsilon^{12} = 1$. We restrict ourselves to quadratic terms in fermions.

2.1 The ungauged case

The graviton multiplet for minimal five-dimensional supergravity consists of the Fünfbein ${e_\mu}^a$, a symplectic Majorana gravitino $\psi_{\mu i}$ and a vector A_μ. The dynamics is governed by the Lagrangian

$$\begin{aligned}\mathcal{L} =& \sqrt{g}\left[-\frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\psi}_\mu{}^i\Gamma^{\mu\nu\rho}D_\nu\psi_{\rho i} - \frac{3}{8\sqrt{6}}i\bar{\psi}_\mu{}^i\left(\Gamma^{\mu\nu\rho\sigma} + 2g^{\mu\nu}g^{\rho\sigma}\right)F_{\nu\rho}\psi_{\sigma i}\right] \\ &+ \frac{1}{6\sqrt{6}}\varepsilon^{\mu\nu\rho\sigma\lambda}A_\mu F_{\nu\rho}F_{\sigma\lambda}\,, \end{aligned} \tag{2.1}$$

where the field strength is given by $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ and D_μ is the covariant derivative with respect to general coordinate and Lorentz transformations.

The action is invariant under ungauged supersymmetry transformations given by

$$\begin{aligned}\delta {e_\mu}^m &= \frac{1}{2}\bar{\epsilon}^i\Gamma^m\psi_{\mu i}\,, \\ \delta\psi_{\mu i} &= D_\mu\epsilon_i + \frac{1}{4\sqrt{6}}i\left({\Gamma_\mu}^{\nu\rho} - 4{\delta_\mu}^\nu\Gamma^\rho\right)F_{\nu\rho}\epsilon_i\,, \\ \delta A_\mu &= -\frac{\sqrt{6}}{4}i\bar{\epsilon}^i\psi_{\mu i}\,, \end{aligned} \tag{2.2}$$

The commutator of two supersymmetry transformations generates the supersymmetry algebra

$$[\delta_1, \delta_2] = \delta_{\text{gct}} + \delta_{\text{Lorentz}} + \delta_{\text{susy}} + \delta_{\text{gauge}} + \delta_{\mathcal{L}}\,, \tag{2.3}$$

with the following parameters for the general coordinate, local Lorentz, supersymmetry and gauge transformations:[3]

$$\begin{aligned}\xi^\mu &= \frac{1}{2}\bar{\epsilon}_1^i\Gamma^\mu\epsilon_{2i}\,, \\ \Lambda^{mn} &= \xi^\nu{\omega_\nu}^{mn} + \frac{1}{4\sqrt{6}}i\bar{\epsilon}_1^i(\Gamma^{mnpq} + 4g^{mp}g^{nq})F_{pq}\epsilon_{2i}\,, \\ \eta^i &= -\xi^\mu\psi_{\mu i}\,, \\ \lambda^{(0)} &= -\frac{\sqrt{6}}{4}i\bar{\epsilon}_1^i\epsilon_{2i} - \xi^\nu A_\nu\,. \end{aligned} \tag{2.4}$$

[3]We differ with respect to [29] in the sign of the third term of the Lorentz transformation.

Here we use the following conventions:

$$\begin{aligned}
\delta_{\rm gct} A_\mu &= -\xi^\nu \partial_\nu A_\mu - A_\nu \partial_\mu \xi^\nu \,, \\
\delta_{\rm Lorentz} e_\mu{}^m &= -\Lambda^m{}_n e_\mu{}^n \,, \\
\delta_{\rm gauge} A_\mu &= -\partial_\mu \lambda^{(0)} \,,
\end{aligned} \tag{2.5}$$

with the obvious generalisation of general coordinate transformations to other forms.

The last term in (2.3) is a possible first-order field equation that can occur when closing the algebra. This is a common feature for the fermions, on which supersymmetry only closes modulo their equations of motion. In the following we will also find first-order constraints when realising the supersymmetry algebra on tensors of higher rank.

In addition to the local symmetries discussed above, the theory also has a global SU(2) R-symmetry. This symmetry only acts on the gravitino (in the fundamental representation) while the metric and vector are invariant under it.

We now would like to see whether one can realise the supersymmetry algebra on other fields as well. We start with a tensor and make the following Ansätze for the transformation under supersymmetry:

$$\delta B_{\mu\nu} = b_1 \bar{\epsilon}^i \Gamma_{[\mu} \psi_{\nu]i} + b_2 A_{[\mu} \delta A_{\nu]} \,. \tag{2.6}$$

One finds that the supersymmetry algebra closes provided $b_1 = \frac{3}{4} b_2 = -\frac{1}{2}\sqrt{6}$ and up to both the gauge transformations

$$\delta_{\rm gauge} B_{\mu\nu} = -2\partial_{[\mu}\lambda^{(1)}_{\nu]} - \frac{1}{3}\sqrt{6}\lambda^{(0)} F_{\mu\nu} \,, \quad \lambda^{(1)}_\nu = -B_{\nu\sigma}\xi^\sigma + \frac{1}{4}\sqrt{6}\bar{\epsilon}^i_1 \Gamma_\nu \epsilon_{2i} - \frac{1}{2} i \bar{\epsilon}^i_1 \epsilon_{2i} A_\nu \,, \tag{2.7}$$

and the duality relation, or first-order field equation,

$$\delta_{\mathcal{L}} B_{\mu\nu} = -\left(H_{\mu\nu\rho} - \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma\lambda} F^{\sigma\lambda} \right) \xi^\rho \,, \quad H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} - \sqrt{6} A_{[\mu} F_{\nu\rho]} \,. \tag{2.8}$$

Since this has to vanish for all supersymmetry transformations we have to require the equation in brackets to vanish. Indeed, from this duality relation follows the field equation for the vector

$$\nabla^\mu F_{\mu\nu} = -\frac{1}{2\sqrt{6}} \sqrt{-g} \varepsilon_{\nu\mu_1 \ldots \mu_4} F^{\mu_1\mu_2} F^{\mu_3\mu_4} \,, \tag{2.9}$$

which can also be derived from the action (2.1). Hence we conclude that it is possible to realise supersymmetry on a tensor, provided it is the Hodge dual to the vector. Summing up, the supersymmetry algebra only closes up to the duality relation (2.8), which can be seen as a bosonic first-order field equation.

Turning to higher-rank anti-symmetric tensors, it can be seen that the algebra only allows for supersymmetry transformations of the form $\bar{\epsilon}^i \Gamma_{[\mu_i \cdots \mu_n} \psi^j_{\mu_{n+1}]}$ which are anti-symmetric in i and j when $n = 0, 1$ and symmetric when $n = 2, 3$ (mod 4). Therefore we make the following Ansätze:

$$
\begin{aligned}
\delta C^{ij}_{\mu\nu\rho} &= i c_1 \bar{\epsilon}^{(i}\Gamma_{[\mu\nu}\psi^{j)}_{\rho]}\,,\\
\delta D^{ij}_{\mu\nu\rho\sigma} &= d_1 \bar{\epsilon}^{(i}\Gamma_{[\mu\nu\rho}\psi^{j)}_{\sigma]} + d_2 A_{[\mu}\delta C^{ij}_{\nu\rho\sigma]}\,,\\
\delta E_{\mu\nu\rho\sigma\tau} &= i e_1 \bar{\epsilon}^{i}\Gamma_{[\mu\nu\rho\sigma}\psi_{\tau]i} + e_2 A_{[\mu}B_{\nu\rho}\delta B_{\sigma\tau]}\,,
\end{aligned}
\tag{2.10}
$$

where the first two lines are symmetric in i and j. Note that we could have included more terms, e.g. $C^{ij}\wedge\delta A$ in δD^{ij}, but these can be absorbed into a redefinition of D^{ij}. The above Ansätze are the most general modulo such redefinitions. In addition we can impose the symplectic reality conditions

$$
C^{ij} - C^{*}_{ij} = D^{ij} - D^{*}_{ij} = 0\,. \tag{2.11}
$$

It can be verified that these conditions are invariant under the above supersymmetry transformations and under the SU(2) R-symmetry. Under the latter these higher-rank tensors therefore transform as triplets.[4] Note that the original bosonic fields (i.e. the metric and the vector) are invariant under the SU(2) symmetry; until the introduction of the higher-rank tensors this is a symmetry that only acts on the fermionic sector of the theory.

The closure of the supersymmetry algebra on these higher-rank tensors requires the following constants:

$$
c_1 d_2 = -\sqrt{6}d_1\,,\quad e_2 = 0\,, \tag{2.12}
$$

and associated gauge transformations with parameters:

$$
\begin{aligned}
\delta_{\text{gauge}}C^{ij}_{\mu\nu\rho} &= -3\partial_{[\mu}\lambda^{(2)ij}_{\nu\rho]}\,, & \lambda^{(2)ij}_{\mu\nu} &= -C^{ij}_{\mu\nu\rho}\xi^{\rho} + \frac{1}{3}ic_1\bar{\epsilon}^{(i}_1\Gamma_{\mu\nu}\epsilon^{j)}_2\,,\\
\delta_{\text{gauge}}D^{ij}_{\mu\nu\rho\sigma} &= -4\partial_{[\mu}\lambda^{(3)ij}_{\nu\rho\sigma]}\,, & \lambda^{(3)ij}_{\mu\nu\rho} &= -D^{ij}_{\mu\nu\rho\sigma}\xi^{\sigma} - \frac{1}{4}d_1\left(\bar{\epsilon}^{(i}_1\Gamma_{\mu\nu\rho}\epsilon^{j)}_2 - \sqrt{6}iA_{[\mu}\bar{\epsilon}^{(i}_1\Gamma_{\nu\rho]}\epsilon^{j)}_2\right)\,,\\
\delta_{\text{gauge}}E_{\mu\nu\rho\sigma\tau} &= -5\partial_{[\mu}\lambda^{(4)}_{\nu\rho\sigma\tau]}\,, & \lambda^{(4)}_{\mu\nu\rho\sigma} &= -E_{\mu\nu\rho\sigma\tau}\xi^{\tau} + \frac{1}{5}ie_1\bar{\epsilon}^{i}_1\Gamma_{\mu\nu\rho\sigma}\epsilon_{2i}\,.
\end{aligned}
\tag{2.13}
$$

In addition, on the right hand side of the supersymmetry algebra appear the following first-order field equations for the three- and four-forms:

$$
\delta_{\mathcal{L}}C^{ij}_{\mu\nu\rho} = -\left(4\partial_{[\mu}C^{ij}_{\nu\rho\sigma]}\right)\xi^{\sigma}\,,\quad \delta_{\mathcal{L}}D^{ij}_{\mu\nu\rho\sigma} = -\left(5\partial_{[\mu}D^{ij}_{\nu\rho\sigma\tau]}\right)\xi^{\tau}\,, \tag{2.14}
$$

which imply that their curvatures vanish, i.e. these potentials are closed. In combination with their gauge transformations this implies that they do not carry any local degrees of freedom.[5] Indeed, they can only be relevant in topologically non-trivial manifolds, e.g. when they are proportional to volume forms of non-contractible cycles.

[4] In the first preprint version of this paper we only considered the trace of these symmetric representations. This is not SU(2) covariant, as was correctly pointed out afterwards in [30]. However, the introduction of the triplet representations above does give an SU(2)-covariant formulation.

[5] A similar phenomenon, gauge vectors with vanishing field strengths and no local degrees of freedom, was encountered in [16] in the context of $d = 2$ supergravity.

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A complementary conclusion can be reached for the five-form E. Its supersymmetry transformation is proportional to that of the Levi-Civita tensor, which is

$$\delta(\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma\tau}) = -\frac{5}{2} i\bar{\epsilon}^i \Gamma_{[\mu\nu\rho\sigma}\psi_{\tau]i}\,, \tag{2.15}$$

and hence E is not an independent field but rather composed of the metric, i.e. it it proportional to the volume form of space-time: $E = -\frac{2}{5} e_1 \varepsilon$. Indeed, with this identification $\lambda^{(4)}$ vanishes automatically, consistent with the absence of a gauge transformation for the Levi-Civita tensor.

Hence there are no local degrees of freedom associated to the potentials C^{ij} and D^{ij} and there is no independent five-form potential E. It is interesting to note that the commutator of two susy transformations on these potentials turns out to be given by a gauge transformation:

$$\begin{aligned}
[\delta_1,\delta_2]C^{ij}_{\mu\nu\rho} &= -3\partial_{[\mu}\tilde{\lambda}^{(2)ij}_{\nu\rho]}\,, & \tilde{\lambda}^{(2)ij}_{\mu\nu} &= \frac{1}{3} i c_1 \bar{\epsilon}_1^{(i}\Gamma_{\mu\nu}\epsilon_2^{j)}\,,\\
[\delta_1,\delta_2]D^{ij}_{\mu\nu\rho\sigma} &= -4\partial_{[\mu}\tilde{\lambda}^{(3)ij}_{\nu\rho\sigma]}\,, & \tilde{\lambda}^{(3)ij}_{\mu\nu\rho} &= -\frac{1}{4} d_1 \left(\bar{\epsilon}_1^{(i}\Gamma_{\mu\nu\rho}\epsilon_2^{j)} - \sqrt{6} i A_{[\mu}\bar{\epsilon}_1^{(i}\Gamma_{\nu\rho]}\epsilon_2^{j)}\right)\,,\\
[\delta_1,\delta_2]E_{\mu\nu\rho\sigma\tau} &= -5\partial_{[\mu}\tilde{\lambda}^{(4)}_{\nu\rho\sigma\tau]}\,, & \tilde{\lambda}^{(4)}_{\mu\nu\rho\sigma} &= \frac{1}{5} i e_1 \bar{\epsilon}_1^i \Gamma_{\mu\nu\rho\sigma}\epsilon_{2i}\,.
\end{aligned} \tag{2.16}$$

One finds that the commutator of supersymmetry does not lead to any terms involving the parameter ξ^μ of general coordinate transformations. These terms cancel separately on the right hand side of the supersymmetry algebra (2.3) due to the contribution (2.14). Hence the supersymmetry algebra (2.3) is realised in a rather trivial way on these potentials. Indeed, due to the above commutators, setting C^{ij} and D^{ij} to zero by the gauge transformations (2.13) is consistent with supersymmetry.

The presence of the triplets of three- and four-forms, on which supersymmetry can be realised provided they have vanishing curvature, may have come as a surprise at this point. In the next subsection we will see however that they are necessary for the inclusion of a gauge coupling constant.

2.2 The gauged case

We now consider the gauging of a U(1) subgroup of the SU(2) R-symmetry group, with coupling constant[6] g [29]. The action for this gauged supergravity is

$$\begin{aligned}
\mathcal{L} =& \sqrt{g}\Big[-\frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\psi}_\mu{}^i\Gamma^{\mu\nu\rho}(D_\nu\psi_{\rho i} - gA_\nu\delta_{ij}\psi^j_\rho) + \\
&\quad - \frac{3}{8\sqrt{6}} i\bar{\psi}_\mu{}^i(\Gamma^{\mu\nu\rho\sigma} + 2g^{\mu\nu}g^{\rho\sigma})F_{\nu\rho}\psi_{\sigma i} - \frac{1}{4}\sqrt{6} i g\bar{\psi}^i_\mu\Gamma^{\mu\nu}\psi^j_\nu\delta_{ij} + 4g^2\Big]\\
&+ \frac{1}{6\sqrt{6}}\varepsilon^{\mu\nu\rho\sigma\lambda}A_\mu F_{\nu\rho}F_{\sigma\lambda}\,,
\end{aligned} \tag{2.17}$$

[6] Here we have chosen a specific embedding of the gauged U(1) in SU(2) without loss of generality. To describe the other embeddings one should replace $g\delta_{ij}$ by g_{ij}, which is symmetric and subject to a symplectic reality condition like (2.11).

where the field strength is still given by $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. These are invariant under the following supersymmetry variations:

$$\begin{aligned}
\delta e_\mu{}^m &= \frac{1}{2}\bar\epsilon^i\Gamma^m\psi_{\mu i}\,,\\
\delta\psi_{\mu i} &= D_\mu\epsilon_i + \frac{1}{4\sqrt6}i(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu\Gamma^\rho)F_{\nu\rho}\epsilon_i - gA_\mu\delta_{ij}\epsilon^j - \frac{1}{\sqrt6}ig\Gamma_\mu\delta_{ij}\epsilon^j\,,\\
\delta A_\mu &= -\frac{\sqrt6}{4}i\bar\epsilon^i\psi_{\mu i}\,.
\end{aligned}\tag{2.18}$$

Note that there are only corrections to the supersymmetry variation of the fermion and not to those of the metric and vector.

It turns out that the supersymmetry variations of all higher-rank potentials are unchanged as well, i.e. equal to their ungauged expressions, just like the other bosons in (2.18). The only differences appear on the right hand side of the supersymmetry algebra for the potentials B and D^{ij}: the two-form gauge transformation becomes

$$\delta_{\text{gauge}}B_{\mu\nu} = -2\partial_{[\mu}\lambda^{(1)}_{\nu]} - \frac{1}{3}\sqrt6\lambda^{(0)}F_{\mu\nu} + \beta g\lambda^{(2)ij}_{\mu\nu}\delta_{ij}\,,\tag{2.19}$$

where $\beta = \sqrt6 b_1/c_1$ and the duality relations (or first-order field equations) for B and D^{ij} become

$$\begin{aligned}
\delta_{\mathcal L}B_{\mu\nu} &= -\left(H_{\mu\nu\rho} - \beta gC^{ij}_{\mu\nu\rho}\delta_{ij} - \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma\lambda}F^{\sigma\lambda}\right)\xi^\rho\,,\\
\delta_{\mathcal L}D^{ij}_{\mu\nu\rho\sigma} &= -\left(5\partial_{[\mu}D^{ij}_{\nu\rho\sigma\tau]} - \frac{1}{2}\gamma g\delta^{ij}E_{\mu\nu\rho\sigma\tau}\right)\xi^\tau\,,
\end{aligned}\tag{2.20}$$

where $\gamma = -\frac{5}{3}\sqrt6 d_1/e_1$.

Note that the trace part of the field strength of the four-forms potential D^{ij} is non-vanishing in the gauged theory. This implies that this potential, unlike in the ungauged case, can no longer be gauged away locally. Recalling the identification of E with the Levi-Civita tensor, the duality relation for the trace of the four-form D^{ij} implies that its field strength is Hodge dual to the mass parameter or gauge coupling constant g. This is analogous to the identification of e.g. the field strength of the nine-form in IIA supergravity [31, 32] with Romans' mass parameter [33]. Hence the presence of the four-forms in the supersymmetry algebra is directly related to the possibility of gauging the U(1) group. This explains why D^{ij} also appeared in the ungauged case. Indeed, its appearance there can be seen as a necessary condition for and hence a prediction of the existence of gauged supergravity.

In the same spirit, the gauging explains the presence of the three-forms C^{ij} in the superalgebra. Their gauge transformations are necessary to be able to realise supersymmetry on the tensor in the gauged case, since the latter transforms under the former. Indeed, the tensor B is pure gauge due to the $\lambda^{(2)ij}$ term in its gauge transformation. When gauging away B, the associated degrees of freedom are carried by the trace of C^{ij}. It has a vanishing field strength but its gauge freedom has been fixed, giving rise to the same number of local degrees of freedom as a two-form gauge potential. Alternatively, we could locally

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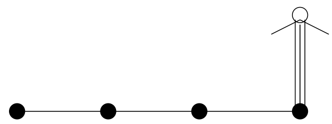

Figure 1: The extended Dynkin diagram of G_2^{+++} with its horizontal A_4 subalgebra.

choose to set C^{ij} to zero, but in order to preserve this gauge choice under the commutator of supersymmetry we need a compensating transformation $\tilde{\lambda}^{(2)ij}$ given by (2.16). Also note that, although the field strength H contains a term $gC^{ij}\delta_{ij}$, this does does not modify the field equation (2.9), in accordance with the above action for the gauged case.

Even though C^{ij} still has vanishing curvature and the commutator of supersymmetry acts as a total derivative on it, the three-forms turns out to play a crucial role in dualising the vector into a tensor when $g \neq 0$. Indeed, it is impossible to realise supersymmetry on the tensor without including C^{ij}. This is in contrast to the ungauged case, where it is consistent to consider only potentials up to a certain rank. In the gauged case such a hierarchy is no longer present: a higher-rank potential can be necessary to realise supersymmetry on a potential of lower rank, as we have found for B and C^{ij}.

Summarising, we have found in this subsection that the presence of C^{ij} and D^{ij} in the supersymmetry algebra are both related to the gauging: the four-forms predict the possibility to include a gauging, while the three-forms are necessary to dualise the vector in the gauged case. The latter seems to be a novel mechanism that we have not encountered in the literature.[7]

3. Very-extended G_2

In this section we will recapitulate the predictions from very extended G_2 and compare against the findings from the supersymmetry algebra.

Given a Kac-Moody algebra which is the very extension of some Lie algebra, one can decompose its adjoint representation into representations of a Lie subalgebra A_n (the 'gravity line'). These are labelled by their level l in the Kac-Moody algebra and can be interpreted to correspond with fields in $d = n + 1$ dimensions. This has been explained in e.g. [23] and references therein, where more details can be found.

The relevant very extended algebra in the present case is G_2^{+++}, whose extended Dynkin diagram is given in the figure. Its decomposition in A_4 representations has been given in [23], from which we copy the relevant table. Note that there is no internal SU(2) symmetry in addition to the space-time A_4 symmetry, and hence all ensuing representations will be singlets of SU(2).

The space-time field interpretation for the first four entries is as graviton, vector and tensor, respectively. These agree with our results in the previous section, where we found

[7]For instance, in the formalism of [34] for gauged $d = 5$ maximal supergravities, dual tensors are also introduced and the supersymmetry algebra only closes up to first-order duality relations for them, but there are no terms like $g\lambda^{(2)}$ in their gauge transformations.

l	A_4 weight	G_2^{+++} element α	α^2	$ht(\alpha)$	μ	Interpretation
0	[1,0,0,1]	(1,1,1,1,0)	2	4	1	graviton
1	[0,0,0,1]	(0,0,0,0,1)	2	1	1	vector A
2	[0,0,1,0]	(0,0,0,1,2)	2	3	1	tensor B
3	[0,0,1,1]	(0,0,0,1,3)	6	4	1	dual graviton
3	[0,1,0,0]	(0,0,1,2,3)	0	6	0	
4	[0,1,0,1]	(0,0,1,2,4)	2	7	1	mixed
4	[1,0,0,0]	(0,1,2,3,4)	-4	10	0	
5	[0,1,1,0]	(0,0,1,3,5)	2	9	1	mixed
5	[1,0,0,1]	(0,1,2,3,5)	-4	11	1	mixed
5	[0,0,0,0]	(1,2,3,4,5)	-10	15	0	

Table 1: The first levels of the decomposition of G_2^{+++} with respect to A_4. All representations are SU(2) singlets.

that one can realise supersymmetry on $e_\mu{}^a$, A_μ and $B_{\mu\nu}$. The fourth should correspond to the dual graviton, which we did not consider since it has mixed symmetries and we restrict ourselves to anti-symmetric tensors. The remaining entries either have mixed symmetries or are absent (with vanishing multiplicity μ). At higher levels $l \geq 6$ there are only representations with more than six space-time indices.

Note in particular that there are no four-form potentials[8] predicted by very extended G_2. This is in clear contradistinction to the results from the supersymmetry algebra, which does allow for a triplet of four-forms whose field strength is dual to the gauge coupling constant. Hence it emerges that very extended G_2 should be associated to ungauged $d = 5$ minimal supergravity and not to the corresponding gauged supergravity. In addition to the absence of the four-forms, there is also no five-form predicted by very extended G_2. This agrees with both the ungauged and the gauged supersymmetry algebra.

Given that G_2^{+++} is associated to the ungauged case, the vector can be identified as a raising operator from which the entire bosonic gauge algebra of the ungauged theory can be generated. To see this one must first make the following redefinition of the gauge algebra. As things stand, the gauge transformation (2.7) is Abelian and non-local, due to the term proportional to F. One can redefine the gauge parameter by $\lambda^{(1)\prime} = \lambda^{(1)} + \frac{1}{3}\sqrt{6}\lambda^{(0)}A$ to obtain the transformation

$$\delta_{\text{gauge}} B_{\mu\nu} = -2\partial_{[\mu}\lambda^{(1)\prime}_{\nu]} + \frac{2}{3}\sqrt{6}\partial_{[\mu}\lambda^{(0)}A_{\nu]}\,, \tag{3.1}$$

which is non-Abelian and local. A similar phenomenon was observed in [21], where the non-Abelian gauge algebra was interpreted in terms of raising operators. In our case these

[8]The same absence was noted by [35] in the context of a one-dimensional σ-model based on overextended G_2. There it was interpreted as predicting the absence of R^2 higher-order corrections, which however do occur for this supergravity. This paradox may be resolved by the observation of [36, 37] that higher-order corrections correspond to weights instead of roots of the overextended algebra.

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are the gauge transformation $\mathbf{1}$ of the vector, and we sketchily have

$$[\mathbf{1}, \mathbf{1}] = \mathbf{2}\,, \tag{3.2}$$

where $\mathbf{2}$ is the gauge transformation of the tensor. Hence the vector can be interpreted as the raising operator $\mathbf{1}$, in agreement with the fact that the node outside of the gravity line is at the outer right position in the extended Dynkin diagram, and all other gauge transformations can be generated by considering multiple commutators of it. For instance, the double commutator $[[\mathbf{1}, \mathbf{1}], \mathbf{1}]$ should give rise to the gauge transformation of the dual graviton, see also [38]. From this point of view it also follows that the multiple commutators of the singlet $\mathbf{1}$ can not give rise to the gauge transformations of the triplets of higher-rank forms.

4. Discussion

In this note we have compared the possibilities to realise the $N = 2$, $d = 5$ supersymmetry algebra on higher-rank tensors with the predictions of very extended G_2. Our main results are the inclusion of triplets of three- and four-forms in the supersymmetry algebra, necessary for the gauging of the U(1), and the failure of very extended G_2 to capture these forms.

The absence of the four-forms in very extended G_2 is in contrast to the previously considered case of E_{11} and gaugings of maximal supergravities, where the very extended algebra contains $(d-1)$-forms corresponding to the possible gauge coupling constants or mass parameters. A caveat here is that there are more deformations allowed for by supersymmetry which are not captured by E_{11}, that correspond to the gauging of the 'trombone' or scale symmetry of the field equations and Bianchi identities [39 – 41]. These are not symmetries of the Lagrangian, and indeed their gauging leads to field equations that cannot be derived from an action principle. In addition, these symmetries are expected to be broken by higher-order corrections. The situation considered here is therefore of a different nature: gauging the U(1) leads to a perfectly bonafide gauged supergravity with an action principle. It does differ from gauged maximal supergravity in that its original bosonic fields are invariant under the symmetry that is gauged, while the gauge groups of maximal supergravity do act on the original bosonic sector [30].

The absence of the U(1) gauging is all the more striking from the following point of view. The $N = 2$ gauged supergravity can be obtained as a truncation of $N = 8$ supergravity with an SO(6) gauging [42], which is included in E_{11}. The gauge coupling constant survives the truncation from $N = 8$ to minimal $N = 2$ pure supergravity. From the very extended algebras point of view, E_{11} can be truncated to very extended G_2. This works flawlessly for the propagating degrees of freedom, but the gauge coupling constant is lost in the process. This suggests that there is a different truncation of E_{11}, which contains both G_2^{+++} and the SU(2) triplets of three- and four-forms generators, and therefore accounts for both propagating and non-propagating degrees of freedom. It would be interesting to uncover whether such an algebra exists and what its structure is.

In this note we have presented an example with eight supercharges and the non-simply laced G_2^{+++}, where the very extended algebra does not capture the possible gauging of the supergravity theory. Note that this is even without including any matter multiplets, which is an additional option in less than maximal supergravity. It will be very interesting to extend this analysis to other cases, with other supergravities and very extended algebras, and to investigate what the requirements or reasons are for the non-propagating degrees of freedom to be present or absent in the very extended algebras. In the latter case, one could also look for possible extensions of these algebras that do contain all non-propagating degrees of freedom, similar to a possible truncation of E_{11} that extends G_2^{+++} with the triplets of generators.

We have also observed that in the gauged case the supersymmetry algebra does not preserve the level structure. That is, the commutator of supersymmetry on a form can receive gauge contributions from a higher-rank form, in our case B and C^{ij}. For this reason it is not always possible to only include fields up to a certain level l. One may expect this to be a general phenomena that will also occur for level decompositions in other theories.

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